Minnesota State High School Mathematics League

2021-22 Meet 5, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 20 minutes for this event.

NO CALCULATORS are allowed on this event. All answers are integers.

______ 1. Each of the numbers from 1 to 6 is entered into the "magic triangle" puzzle shown in *Figure 1* so that the three numbers along each edge add to the same value, its "magic sum". Three numbers have already been entered. What is the magic sum for this triangle?

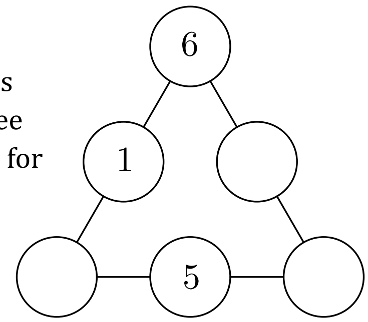


Figure 1

______ 2. Out of 135 students surveyed, 28 are able to juggle and 42 can do a handstand. If 88 students are unable to do either skill, how many students are able to juggle and can also do a handstand?

______ 3. The addition puzzle shown in *Figure 3* represents the valid sum OLD + OLD + SEÑOR = MAYOR, where each letter represents a different digit, and all instances of a particular letter correspond to the same digit. What 5-digit number is represented by the word MAYOR?

```
      O L D
      O L D
+ S E Ñ O R
-----------
  M A Y O R
```

Figure 3

______ 4. Determine the **least** possible value of $x \cdot y$ among all real solutions (x, y, z) to the system:

$$x + y + z = 16$$
$$x^2 + y^2 + z^2 = 100$$

Name: ______________ Team: ______________

Minnesota State High School Mathematics League

2021-22 Meet 5, Individual Event A

SOLUTIONS

11

1. Each of the numbers from 1 to 6 is entered into the "magic triangle" puzzle shown in *Figure 1* so that the three numbers along each edge add to the same value, its "magic sum". Three numbers have already been entered. What is the magic sum for this triangle?

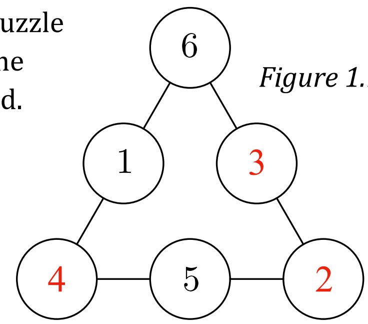


Figure 1.1

The remaining numbers 2, 3, and 4 have yet to be placed; some experimentation leads to the placement shown in Figure 1.1, which has a magic sum of 11 (= 6 + 1 + 4 = 4 + 5 + 2 = 2 + 3 + 6).

23

2. Out of 135 students surveyed, 28 are able to juggle and 42 can do a handstand. If 88 students are unable to do either skill, how many students are able to juggle and can also do a handstand?

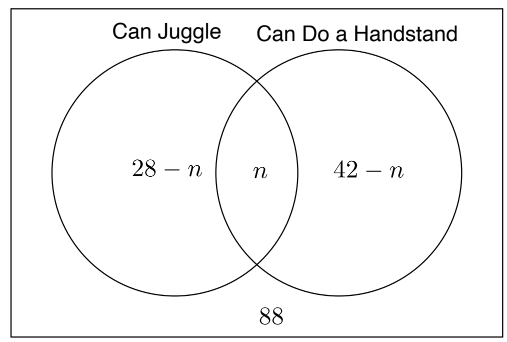


Figure 2.1

Consider the Venn Diagram representation shown in Figure 2.1. If n students can do both, then $28 - n$ can juggle (but not do a handstand), and $42 - n$ can do a handstand (but not juggle). Since 88 can't do either skill, and there are 135 students in all:

$$135 = (28 - n) + n + (42 - n) + 88$$
$$\implies 135 = 158 - n \implies n = 23$$

31486

3. The addition puzzle shown in *Figure 3* represents the valid sum OLD + OLD + SEÑOR = MAYOR, where each letter represents a different digit, and all instances of a particular letter correspond to the same digit. What 5-digit number is represented by the word MAYOR?

Figure 3.1

```
      O L D
      O L D
+ S E Ñ O R
-----------
  M A Y O R
```

Considering the last two digits, we must have <u>L</u><u>D</u>=50, so L=5 and D=0. And now since O+O+<u>S</u><u>E</u><u>N</u>+1=<u>M</u><u>A</u><u>Y</u>, we must have that M = S + 1 and since D = 0, we have E = 9 and A = 1. We are left with O+O+<u>S</u><u>9</u><u>N</u>+1=<u>M</u><u>1</u><u>Y</u>, so O+O+N+1=20+Y. The digits we have left are 2, 3, 4, 6, 7, 8 and with the additional restraint that M = S + 1, we get that O = 8 and N = 7, so Y = 4 and S = 2, M = 3. Finally, the last letter R = 6, so MAYOR is represented by 31486.

14

4. Determine the **least** possible value of $x \cdot y$ among all <u>real</u> solutions (x, y, z) to the system:

$$x + y + z = 16$$
$$x^2 + y^2 + z^2 = 100$$

Notice that $2xy = (x + y)^2 - (x^2 + y^2) = (16 - z)^2 - (100 - z^2) = 156 - 32z + 2z^2$.
Then $xy = z^2 - 16z + 78 = (z - 8)^2 + 14$, *so since z is real the minimum value of xy is 14.*
(To be complete, verify this minimum can be achieved, e.g. when $(x, y, z) = (4 + \sqrt{2}, 4 - \sqrt{2}, 8)$*).*

Minnesota State High School Mathematics League

2021-22 Meet 5, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event. All answers are integers.

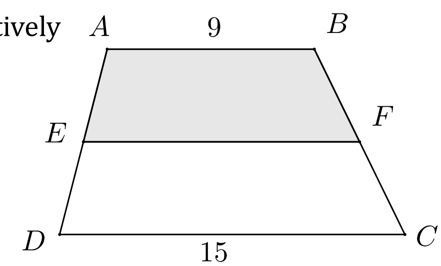
__________ miles

1. The distance from Hutchinson to Monticello on a map is 16 inches. The scale on the map is 1 inch : 2.5 miles. What is the actual distance between Hutchinson and Monticello, in miles?

$m + n =$ __________

2. In *Figure 2*, trapezoid $ABCD$ has parallel sides $\overline{AB}$ of length 9 and $\overline{CD}$ of length 15. Trapezoid $ABFE$ is shaded, where E is the midpoint of $\overline{AD}$ and F is the midpoint of $\overline{BC}$. The fraction of the area of trapezoid $ABCD$ which is shaded can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m + n$.

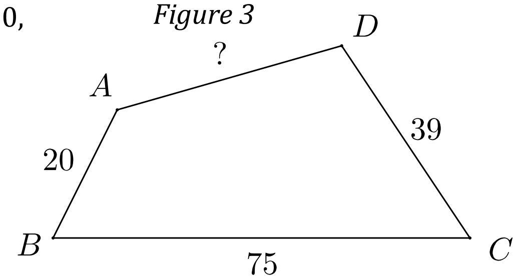
Figure 2



$AD =$ __________

3. In *Figure 3* with quadrilateral $ABCD$ shown, $AB = 20$, $BC = 75$, $CD = 39$, $\cos\angle B = 0.6$, and $\tan\angle C = 2.4$. Determine the length of $\overline{AD}$.

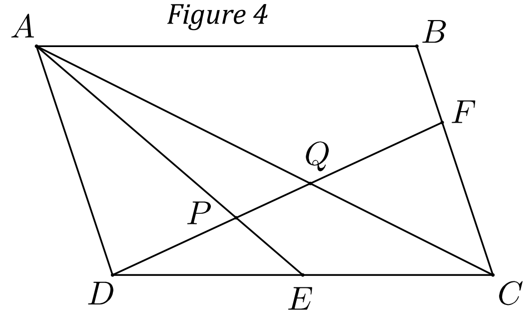
Figure 3



$DF =$ __________

4. *Figure 4* shows parallelogram $ABCD$ with $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$. E is on $\overline{CD}$ and F is on $\overline{BC}$, with $\overline{AE}$ and $\overline{AC}$ intersecting $\overline{DF}$ at P and Q respectively. If $DE = EC = 64$, $BF = 26$, $CF = 52$, and $PQ = 27$, determine the length of $\overline{DF}$.

Figure 4



Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2021-22 Meet 5, Individual Event B

SOLUTIONS

40

1. The distance from Hutchinson to Monticello on a map is 16 inches. The scale on the map is 1 inch : 2.5 miles. What is the actual distance between Hutchinson and Monticello, in miles?

If 1 inch represents 2.5 miles on the map, then 16 inches represents $16 \times 2.5 = 40$ miles on the map, so the actual distance between Hutchinson and Monticello is 40 miles.

23

2. In *Figure 2*, trapezoid $ABCD$ has parallel sides $\overline{AB}$ of length 9 and $\overline{CD}$ of length 15. Trapezoid $ABFE$ is shaded, where E is the midpoint of $\overline{AD}$ and F is the midpoint of $\overline{BC}$. The fraction of the area of trapezoid $ABCD$ which is shaded can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m+n$.

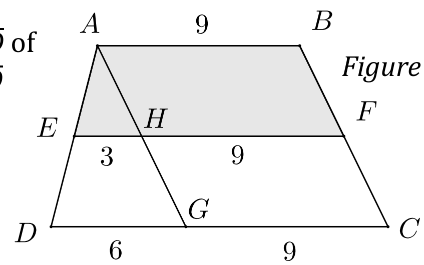


Figure 2.1

Intuition tells us that since E and F are midpoints, then the length of $\overline{EF}$ is 12 (the average of 9 and 15). To prove this, select point G on $\overline{CD}$ so that ABCG is a parallelogram, with $\overline{AG}$ and $\overline{EF}$ meeting at H, as shown in Figure 2.1. Then $\triangle AEH \sim \triangle ADG$, so $\overline{EH} : \overline{DG} = \overline{AE} : \overline{AD} = 1:2$ $\implies \overline{EH} = 3 \implies \overline{EF} = 12$.

Then if ABCD has height h, its area is $\frac{1}{2}(9+15)\,h = 12h$, and ABFE has height $\frac{1}{2}h$ and area $\frac{1}{2}(9+12)\cdot\frac{1}{2}h = \frac{21}{4}h$, so the fraction of ABCD which is shaded is $\frac{\frac{21}{4}}{12} = \frac{21}{48} = \frac{7}{16}$. 7+16=23.

52

3. In *Figure 3* with quadrilateral $ABCD$ shown, $AB = 20$, $BC = 75$, $CD = 39$, $\cos\angle B = 0.6$, and $\tan\angle C = 2.4$. Determine the length of $\overline{AD}$.

Drop altitudes $\overline{AE}$, $\overline{DF}$, and $\overline{AG}$ as shown in Figure 3.1, forming right triangles ABE, CDF, ADG, and rectangle AEFG. In triangle ABE,
$\cos B = \frac{\overline{BE}}{\overline{AB}} = 0.6$, so $\overline{BE} = 12$, and since ABE is a scaled 3-4-5 triangle (or by the Pythagorean Theorem), $\overline{AE} = 16$. Now in triangle CDF,
$\tan C = \frac{\overline{DF}}{\overline{CF}} = 2.4 = \frac{12}{5}$, so CDF is a scaled 5-12-13 triangle, giving $\overline{CF} = 15$, $\overline{DF} = 36$, and so $\overline{EF} = \overline{AG} = 48$ and $\overline{DG} = 20$. Finally ADG is another scaled 5-12-13 triangle so $\overline{AD} = 52$.

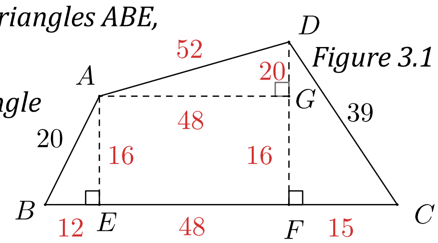


Figure 3.1

120

4. *Figure 4* shows parallelogram $ABCD$ with $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$. E is on $\overline{CD}$ and F is on $\overline{BC}$, with $\overline{AE}$ and $\overline{AC}$ intersecting $\overline{DF}$ at P and Q respectively. If $DE = EC = 64$, $BF = 26$, $CF = 52$, and $PQ = 27$, determine the length of $\overline{DF}$.

Extend $\overrightarrow{AE}$ and $\overrightarrow{BC}$ to meet at R as shown in Figure 4.1, and suppose $DF = x$.
Then $\triangle AQD \sim \triangle CQF \implies QF : QD = 2:3$, so $QD = \frac{3}{5}x$. Similarly since $\triangle APD \sim \triangle RPF$ and $AD = CR$ (from $\triangle ADE \cong \triangle RCE$),
$PF : PD = 5:3$, so $PD = \frac{3}{8}x$. But $PQ = QD - PD = 27$, so $\frac{3}{5}x - \frac{3}{8}x = 27$
$\implies x = 120$.

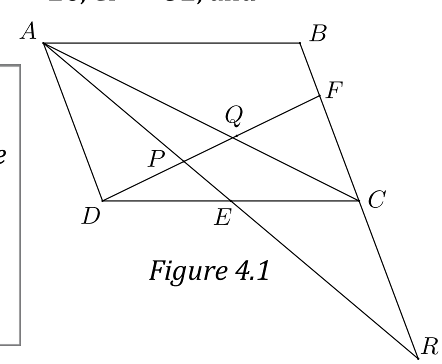


Figure 4.1

Minnesota State High School Mathematics League

2021-22 Meet 5, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event. All answers are integers.

$m + n =$ ________

1. A fair coin is tossed three times. The probability of getting exactly two heads is $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m + n$.

____

2. Andrew, Bailin, Carol, and Dariyah always sit together in the back row at the movie theater. In how many ways can they be seated so that Andrew and Bailin sit next to each other?

____

3. Harrison has 14 identical lollipops that he wishes to give to Madeline, Sophia, and Jia, giving at least 1 to Madeline, at least 2 to Sophia, and at least 3 to Jia. In how many ways can he distribute his lollipops to these three people?

$m + n =$ ________

4. Four numbers a, b, c, and d are chosen at random with replacement from the set $S = \{1,2,3,4,5,6\}$. The probability that lines $ax + by = 1$ and $cx + dy = 1$ intersect in exactly one point can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m + n$.

Name: ________________ Team: ________________

Minnesota State High School Mathematics League

2021-22 Meet 5, Individual Event C

SOLUTIONS

11 1. A fair coin is tossed three times. The probability of getting <u>exactly</u> two heads is $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m+n$.

In this situation it's not too hard to list all cases, which are equally likely: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT. Exactly two heads are achieved in 3 of these 8 cases (HHT, HTH, THH), so the probability is $\frac{3}{8}$. 3+8=11.
More generally, since each coin flip has two possible outcomes, there are $2^3=8$ possibilities when flipped three times. Among the three flips, there are $\binom{3}{2}=3$ ways to select which two came up heads, again for a probability of $\frac{3}{8}$.

12 2. Andrew, Bailin, Carol, and Dariyah always sit together in the back row at the movie theater. In how many ways can they be seated so that Andrew and Bailin sit next to each other?

Since Andrew and Bailin sit together, we initially consider them as a single unit: seating {Andrew/Bailin}, Carol, and Dariyah from left to right can be done in 3!=6 ways. In each case, there are two ways to arrange Andrew and Bailin: either Andrew sits to the left or Bailin does. Overall this gives a total of $6\times 2=12$ seating arrangements.

45 3. Harrison has 14 identical lollipops that he wishes to give to Madeline, Sophia, and Jia, giving at least 1 to Madeline, at least 2 to Sophia, and at least 3 to Jia. In how many ways can he distribute his lollipops to these three people?

Start by giving 1 lollipop to Sophia and 2 to Jia; then the remaining 11 lollipops must be distributed to Madeline, Sophia, and Jia such that each gets at least 1 more lollipop. This is a "stars and bars" counting problem: placing the 11 lollipops in a row (the "stars") there are 10 spaces between them, so we choose two of these spaces as dividers (the "bars") and distribute the first group to Madeline, the second to Sophia, and the third to Jia. There are $\binom{10}{2}=\frac{10\times 9}{2\times 1}=45$ ways to do this, so there are 45 ways to distribute the 14 lollipops with Madeline getting at least 1, Sophie at least 2, and Jia at least 3.

1253 4. Four numbers a, b, c, and d are chosen at random <u>with replacement</u> from the set $S=\{1,2,3,4,5,6\}$. The probability that lines $ax+by=1$ and $cx+dy=1$ intersect in <u>exactly</u> one point can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m+n$.

Two lines intersect in exactly one point precisely when their slopes are different. The slope of $ax+by=1$ is $-\frac{a}{b}$, so the table at right shows slopes for all 36 possibilities (which are equally likely). We can now solve the problem by considering slopes of each line. If the first has slope -1 (6 ways), there are 30 options for a different 2nd slope -> 180 cases. If the first has slope -2 or -1/2 (3 ways each), there are 33 options for a different 2nd slope -> 198 cases total. If the first has slope -3 or -1/3 or -3/2 or -2/3 (2 ways each), there are 34 options for a different 2nd slope -> 272 cases total. For the remaining 16 cases, there are 35 options for a different 2nd slope -> 560 cases total. Overall, the probability of intersecting lines is (180+198+272+560)/6^4=1210/1296=605/648. 605+648=1253.

	a=1	a=2	a=3	a=4	a=5	a=6
b=1	-1	-2	-3	-4	-5	-6
b=2	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2	$-\frac{5}{2}$	-3
b=3	$-\frac{1}{3}$	$-\frac{2}{3}$	-1	$-\frac{4}{3}$	$-\frac{5}{3}$	-2
b=4	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{3}{4}$	-1	$-\frac{5}{4}$	$-\frac{3}{2}$
b=5	$-\frac{1}{5}$	$-\frac{2}{5}$	$-\frac{3}{5}$	$-\frac{4}{5}$	-1	$-\frac{6}{5}$
b=6	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{5}{6}$	-1

Minnesota State High School Mathematics League

2021-22 Meet 5, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event. All answers are integers.

______ *inches*^{2} 1. Kaitlyn has a 5×8 index card. If she shortens the length of one side of this card by 2 inches, the card would have area 30 square inches. What would the area of the card be in square inches if instead she shortens the length of the perpendicular side by 2 inches?

______ 2. The largest prime factor of 20736 is 3 because $20736 = 12^4$. What is the greatest prime number that is a divisor of 20735?

$AB =$ ______ 3. Consider two concentric circles of radius 14 and 22. A chord $\overline{AB}$ of the larger circle passes through the smaller circle, forming chord $\overline{CD}$. If $AB = 3 \cdot CD$, what is the length of $\overline{AB}$?

$N =$ ______ 4. Monique rolls 5 standard 6-sided dice simultaneously and calculates the product of the 5 numbers obtained. The probability that the product is divisible by 6 is $\frac{N}{7776}$. What is N?

Name: ______________ Team: ______________

Minnesota State High School Mathematics League

2021-22 Meet 5, Individual Event D

SOLUTIONS

24

1. Kaitlyn has a 5×8 index card. If she shortens the length of one side of this card by 2 inches, the card would have area 30 square inches. What would the area of the card be in square inches if instead she shortens the length of the perpendicular side by 2 inches?

[2021 Fall AMC 12A, problem #2]

Shortening one side of the card by 2 inches leads to cards which are either 3×8 (area 24) or 5×6 (area 30). Since the card had area 30, it was 5 by 6, so if she'd instead shortened the other side by 2 inches it would have been 3 by 8 and had area 24.

29

2. The largest prime factor of 20736 is 3 because $20736 = 12^4$. What is the greatest prime number that is a divisor of 20735?

[2021 Fall AMC 12B, problem #6]

We can factor as a difference of squares: $12^4 - 1 = (12^2 - 1)(12^2 + 1) = 143 \times 145$ $= (11 \times 13) \times (5 \times 29)$. *The largest prime factor of* 20735 *is therefore* 29.

36

3. Consider two concentric circles of radius 14 and 22. A chord $\overline{AB}$ of the larger circle passes through the smaller circle, forming chord $\overline{CD}$. If $AB = 3 \cdot CD$, what is the length of $\overline{AB}$?

[2021 Fall AMC 12A, problem #11]

Label the common center O, draw line OA intersecting the smaller circle at E and F, and let $CD = x$ (see Figure 3.1). Then by the Power of a Point, using the smaller circle and point A, we have $AC \cdot AD = AE \cdot AF \implies x \cdot 2x = 8 \cdot 36 \implies x = 12$, which in turn gives $AB = 3 \cdot 12 = 36$. This can also be solved using the Pythagorean theorem.

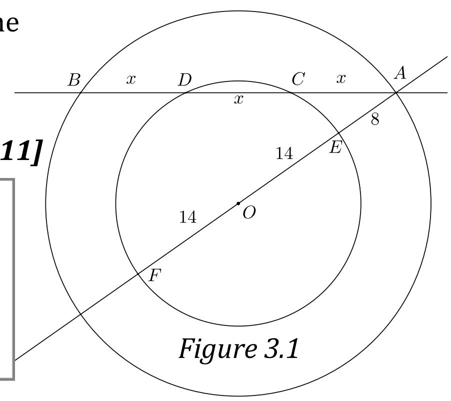


Figure 3.1

6541

4. Monique rolls 5 standard 6-sided dice simultaneously and calculates the product of the 5 numbers obtained. The probability that the product is divisible by 6 is $\frac{N}{7776}$. What is N?

[2021 Fall AMC 12B, problem #11]

Using complementary counting, we first consider the number of ways to form a product <u>not</u> divisible by 6, which has two (overlapping) cases:

<u>Case 1:</u> None of the numbers are divisible by 2 (i.e. each is 1, 3, or 5) $\implies 3^5 = 243$ *possibilities*

<u>Case 2:</u> None of the numbers are divisible by 3 (i.e. each is 1, 2, 4, 5) $\implies 4^5 = 1024$ *possibilities*

Every situation covered by either of these cases gives a product which is not divisible by 6, and every situation not covered by either case must be divisible by both 2 and 3 (and hence 6), so these two cases combine to enumerate all possibilities exactly.

Cases 1 and 2 overlap in the $2^5 = 32$ *cases where each of the numbers is 1 or 5.*

Combining this gives $243 + 1024 - 32 = 1235$ *ways to form a product <u>not</u> divisible by 6, so the probability of getting a product which <u>is</u> divisible by 6 is* $\frac{6^5 - 1235}{6^5} = \frac{6541}{7776}$, *and* $N = 6541$.

Minnesota State High School Mathematics League

2021-22 Meet 5, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 30 minutes, submit only one set of answers. Place your answer to each question on the line provided.

All answers are integers.

__________ 1. How many integer values of x satisfy $|x| < \sqrt{2022}$?

$m + n =$ __________ 2. The probability that an integer, chosen at random between 1 and 99 inclusive, will be divisible by 7 or 9 (or both) can be written as $\frac{m}{n}$ where m and n are relatively prime. What is $m + n$?

$AC =$ __________ 3. Trapezoid $ABCD$ shown in *Figure 3* has $AD \parallel BC$. If $AB = 26$, $BC = 32$, $AD = 16$, and diagonal $BD = 30$, determine the length of the other diagonal $\overline{AC}$.

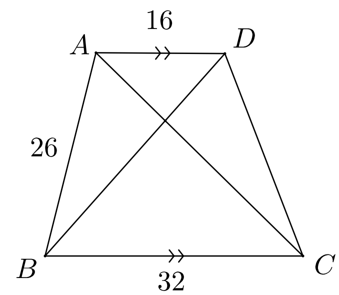


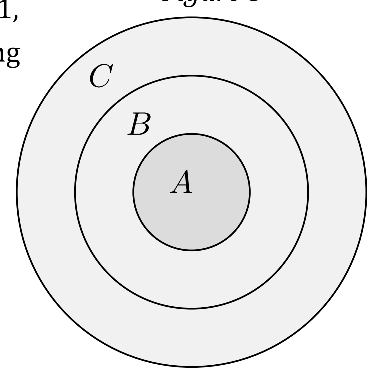
Figure 3

__________ 4. Seven people are standing in a line in order of their age. The leftmost person is 3 years old and the rightmost person is 45 years old. Everyone else's age is <u>one more</u> than the average age of their immediate neighbors. What is the age of the person in the middle of the line (i.e. the person with three people on either side)?

$m + n =$ __________ 5. In *Figure 5*, a target consists of three concentric circles of radius 1, 2, and 3. Darts landing in region A receive 3 points, in circular ring B receive 2 points, and in circular ring C receive 1 point. The probability of scoring <u>exactly</u> 6 points from three darts that randomly hit the target can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m + n$.

(Note that the probability of hitting each ring varies with the area of that ring; they are <u>not</u> equally likely!)

Figure 5



__________ 6. A deck consists of cards numbered 1 through 2022 in order, with 1 on top. The dealer repeatedly discards the top card and moves the next card from the top to the bottom of the deck, continuing until only a single card remains. So the sequence of discarded cards is 1, 3, 5, 7, . . ., 2019, 2021, 2, 6, 10, . . ., and so on. What is the number on the last remaining card?

Team: ____________________

Minnesota State High School Mathematics League

2021-22 Meet 5, Team Event

SOLUTIONS (page 1)

89	1.	How many integer values of x satisfy $\lvert x \rvert < \sqrt{2022}$?
41	2.	The probability that an integer, chosen at random between 1 and 99 inclusive, will be divisible by 7 or 9 (or both) can be written as $\frac{m}{n}$ where m and n are relatively prime. What is $m + n$?
42	3.	Trapezoid $ABCD$ shown in *Figure 3* has $AD \parallel BC$. If $AB = 26$, $BC = 32$, $AD = 16$, and diagonal $BD = 30$, determine the length of the other diagonal $\overline{AC}$.

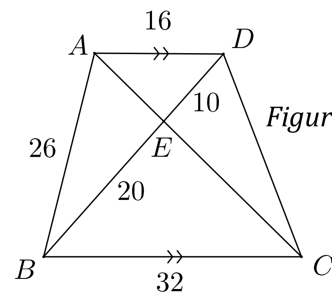
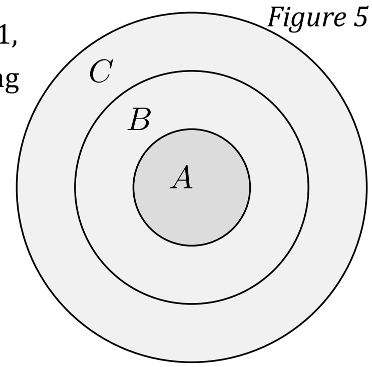


Figure 3.1

33	4.	Seven people are standing in a line in order of their age. The leftmost person is 3 years old and the rightmost person is 45 years old. Everyone else's age is <u>one more</u> than the average age of their immediate neighbors. What is the age of the person in the middle of the line (i.e. the person with three people on either side)?
94	5.	In *Figure 5*, a target consists of three concentric circles of radius 1, 2, and 3. Darts landing in region A receive 3 points, in circular ring B receive 2 points, and in circular ring C receive 1 point. The probability of scoring <u>exactly</u> 6 points from three darts that randomly hit the target can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m + n$. *(Note that the probability of hitting each ring varies with the area of that ring; they are <u>not</u> equally likely!)*

Figure 5



1996	6.	A deck consists of cards numbered 1 through 2022 in order, with 1 on top. The dealer repeatedly discards the top card and moves the next card from the top to the bottom of the deck, continuing until only a single card remains. So the sequence of discarded cards is 1, 3, 5, 7, . . ., 2019, 2021, 2, 6, 10, . . ., and so on. What is the number on the last remaining card?

Minnesota State High School Mathematics League

2021-22 Meet 5, Team Event

SOLUTIONS (page 2)

1. *Note that* $44 < \sqrt{2022} < 45$*, so the integer solutions to* $|x| < \sqrt{2022}$ *are* $-44 \leq x \leq 44$ *which has 89 possibilities: -44, -43, -42, . . ., -1, 0, 1, . . ., 42, 43, or 44.*

2. *7, 14, 21, ..., 91, 98 are divisible by 7 (14 such numbers), and 9, 18, 27, ..., 90, 99 are divisible by 9 (11 such numbers). 63 is on both lists, so there are a total of 14+11-1=24 numbers which are divisible by 7, 9, or both. With 99 numbers to choose from, the probability that a randomly selected number is divisible by 7, 9, or both is* $\frac{24}{99} = \frac{8}{33}$*. 8+33=41.*

3. *Suppose the diagonals intersect at point E. Then* $\triangle ADE \sim \triangle CBE$*, so since BD=30, we must have BE=20 and DE = 10. We can now apply Stewart's Theorem to triangle ABD and cevian AE:*
$$16^2 \cdot 20 + 26^2 \cdot 10 = 30\left(AE^2 + 10 \cdot 20\right) \implies 512 + 676 = 3(AE^2 + 200) \implies 1188 = 3AE^2 + 600$$
$\implies AE^2 = 196 \implies AE = 14$*. Then the same similar triangles as before give* $EC = 28$*, so* $AC = 42$*.*

4. *If three consecutive people are a, b, and c years old, then* $b = \frac{a+c}{2} + 1 \implies c = 2b - a - 2$*. Now suppose the second from left person is x years old. Then using this formula the 3rd from left* $2x - 3 - 2 = 2x - 5$*. Similarly the 4th from left is* $2(2x-5) - x - 2 = 3x - 12$*, the 5th from left is* $4x - 21$*, the 6th from left is* $5x - 32$*, and the 7th from left is* $6x - 45$*. But the 7th from left person is the rightmost, who is 45 years old, so* $6x - 45 = 45$ $\implies x = 15$*. Then the ages are 3, 15, 25, 33, 39, 43, 45, and the middle person is 33 years old.*

5. *To achieve a score of exactly 6, either ring B was hit all three times, or each of the three rings was hit once (which can occur in any of 3!=6 orders).*

   The probabilities of hitting an individual ring with the toss of a single dart are:
$P(A) = \frac{\pi \cdot 1^2}{\pi \cdot 3^2} = \frac{1}{9}$, $P(B) = \frac{\pi \cdot 2^2 - \pi \cdot 1^2}{\pi \cdot 3^2} = \frac{1}{3}$, *and* $P(C) = \frac{\pi \cdot 3^2 - \pi \cdot 2^2}{\pi \cdot 3^2} = \frac{5}{9}$.

   The probability of hitting B three times is $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$*.*

   The probability of hitting ring A, then ring B, then ring C is $\frac{1}{9} \cdot \frac{1}{3} \cdot \frac{5}{9} = \frac{5}{243}$*, and similarly for the other 5 orders, giving a probability of* $\frac{30}{243} = \frac{10}{81}$ *that each target is hit once (in some order).*

   Since these are exclusive, the probability of scoring exactly 6 points is $\frac{1}{27} + \frac{10}{81} = \frac{13}{81}$*. 13+81=94.*

6. *In the general case, starting with a deck numbered 1, 2, 3, . . ., n in order, let* $f(n)$ *denote the last card remaining after this process is applied repeatedly. So for instance* $f(1) = 1, f(2) = 2, f(3) = 2$*, and so on. We'll determine a recursive relationship for* $f(n)$*.*

   First consider $f(2k)$*. After k steps, we've removed the odd cards and are left with 2, 4, 6, ..., 2k: the last card here is twice the last card if we'd been left with 1, 2, 3, . . ., k, so* $f(2k) = 2f(k)$*.*

   Next consider $f(2k+1)$*. After k steps, we're left with 2k+1, 2, 4, 6, ..., 2k, which has the same last card as 0, 2, 4, 6, ..., 2k, twice the last card as 0, 1, 2, 3, ..., k, which is* $f(k+1) - 1$*. So* $f(2k+1) = 2f(k+1) - 2$*.*

   These reduction formulas allow us to determine that $f(2022) = 2f(1011) = 2\left(2f(506) - 2\right) = 4f(506) - 4$ $= 8f(253) - 4 = 16f(127) - 20 = 32f(64) - 52 = 64f(32) - 52 = 128f(16) - 52 = \ldots = 2048f(1) - 52$ $= 2048 - 52 = 1996$*. So the last remaining card is numbered 1996.*

   (Can you find a non-recursive expression for f(n)? This is a version of the "Josephus Problem", the general version of which has every m^{th} *card removed; closed form expressions are only known for m=2 and m=3).*

Minnesota State High School Mathematics League

2021-22 Meet 5 Answers

Event A:

1. 11
2. 23
3. 31486
4. 14

Event B:

1. 40
2. 23
3. 52
4. 120

Event C:

1. 11
2. 12
3. 45
4. 1253

Event D:

1. 24
2. 29
3. 36
4. 6541

Team Event:

1. 89
2. 41
3. 42
4. 33
5. 94
6. 1996